Lesson 21. The Chain Rule

1 This lesson

- Two special cases of the chain rule
- Tree diagrams and the general version of the chain rule

2 Case 1

- Let $z = f(x, y)$ be a function of 2 variables
- Let x and y be functions of 1 variable: $x = g(t)$ and $y = h(t)$

  $\Rightarrow$ z is indirectly a function of t: $z = f(g(t), h(t))$
- Can we find the derivative of z with respect to t?
- **Chain rule (Case 1)**:

Example 1. Let $z = xy - x^2y$, $x = t^2 + 1$, and $y = t - 1$. Find dz/dt.

Example 2. Let $z = x^2y + 3xy^4$, $x = \sin 2t$, and $y = \cos t$. Find dz/dt when $t = 0$.

Example 3. A person's body-mass index (BMI) is given by $B(h, w) = 700w/h^2$, where h is the person's height in inches and w is the person's weight in pounds.

Suppose MIDN Slim is currently 70 inches tall and 140 pounds. MIDN Slim is growing at a rate of 0.1 inches per year, and is gaining weight at a rate of 2 pounds per year. Find the rate at which MIDN Slim's BMI is changing per year.

3 Case 2

- Let $z = f(x, y)$ be a function of 2 variables
- Let x and y be functions of 2 variables: $x = g(s, t)$ and $y = h(s, t)$

  $\Rightarrow$ z is indirectly a function of s and t: $z = f(g(s, t), h(s, t))$
- We can find the derivative of z with respect to s and t
- **Chain rule (Case 2)**:

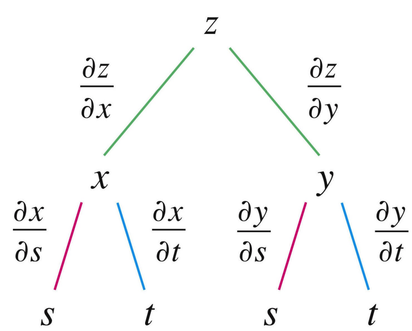
Example 4. Let $z = \sin x \cos y$, $x = st^2$, $y = s^2 t$. Find $\partial z/\partial s$ and $\partial z/\partial t$.

Example 5. Suppose f is a differentiable function of x and y, and $g(u, v) = f(e^u + \sin v, e^u + \cos v)$. Use the following table to compute $g_u(0, 0)$ and $g_v(0, 0)$.

	f	g	f_x	f_y
$(0, 0)$	1	4	8	0
$(1, 2)$	4	1	3	6

4 Tree diagrams

- How can we remember the more complex chain rule (i.e. Case 2)?
- Draw a **tree diagram**:



- To get $\partial z/\partial s$, follow all the paths from z to s:
- This idea can be extended in general to functions of 3 or more variables

Example 6. Write out the chain rule for the case where $z = f(w, x, y)$, $w = g(s, t)$, $x = h(s, t)$, $y = \ell(s, t)$.

Example 7. Write out the chain rule for the case where $w = f(x, y, z)$, $x = g(t)$, $y = h(t)$, $z = \ell(t)$.

Example 8. Let $w = \ln(\sqrt{x^2 + y^2 + z^2})$, $x = \sin t$, $y = \cos t$, $z = t$. Find dw/dt.

Example 9. The length ℓ, width w, and height h of a box change with time. At a certain instant the dimensions are $\ell = 1$ m, $w = 2$ m, and $h = 2$ m. ℓ and w are increasing at a rate of 2 m/s while h is decreasing at a rate of 3 m/s. Find the rate at which the length of the diagonal is changing at that instant.